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Articles

TRANSIENT DYNAMICS AND SCALING PHENOMENA IN URBAN GROWTH

SUSANNA C. MANRUBIA
Fritz Haber Institut der Max Planck Gesellschaft, Faradayweg 4-6, 14195 Berlin, Germany

DAMIÁN H. ZANETTE
Centro Atómico Bariloche, 8400 Bariloche, Argentina

RICARD V. SOLÉ
Complex Systems Research Group, Department of Physics FEN-UPC Campus Nord B5, 08034 Barcelona, Spain
Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA

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Abstract

Urban centers present all over the world striking similarities which translate into universal laws describing their growth and morphology. In this paper, we study a simple two-dimensional cellular automata model containing what we identify as essential ingredients in the demographic change. The slow addition of population to an initially empty area (mimicking migration) added with a reaction diffusion process (representing reorganization of the population) allow us to recover many real data and provide us with a better understanding of the main processes forming and shaping the modern metropolis.

1. INTRODUCTION

Our world has undergone an explosive growth of the urban population in the last decades. This trend to urbanization continues; in fact, many of the cities projected to be megacities in the next century will experience growth rates of about 3% per year in the next decade. This global increase is the result of two mechanisms: One is the internal growth of the city, i.e. the excess of births over deaths (also known as natural growth) and the other is migration.[1] While natural increase becomes more important at higher levels of urbanization, migration is the principal force at the first stages. During the whole formation of the city, the transport of population from one site to another or the decision to establish oneself in this or that area, shape the small and large scale features of the urban center. The history of each individual city provides it with certain patterns, with some uses of the land, with its particular character and always with a personal integration of the past into the present (that successive cultural and political stages reshape and reinterpret). An observer embedded in the bulk of a city will no doubt find striking differences among cities in North and South America, in Europe and in Africa. Nevertheless, if we forget about that two-dimensional embedding and move to the larger scale, those differences transform into amazing and challenging similarities.

In fact, despite the clear differences in their small-scale "details," urban settlements show well-defined generic features that strongly suggest the existence of common universal laws of city growth and morphological organization. One of the best known is Zipf's law[2] which states that the fraction of cities $f(n)$ with n inhabitants shows a definite power-law dependence $f(n) \propto n^{-r}$ with $r \approx 2$. In its original form, this law represents the population $P(R)$ of cities as a function of the rank R of the city: $R = 1$ is assigned to the largest city, $R = 2$ to the second largest, and so on. Remarkably enough, the observed dependence $P(R) \approx R^{-1}$ does not depend on cultural, social or historical factors or on short and long-term economic or political plans. In his book "*Human Behavior and the Principle of Least Effort*,"[2] Zipf reports about communities of 2500 or more inhabitants for the USA in the period 1790–1930. The net population grew extremely fast in that period, but the profile of the function $P(R)$ was always maintained.

Also, the distribution $f(a)$ of the fraction of cities with area a seemingly presents a universal shape $f(a) \propto a^{-s}$ with $s \approx 1.85$.[3] This observation can be put in correspondence with Zipf's law if one considers how the population of a city grows with the increase in the urbanized area. In fact, field studies have led to the so-called *population-area law*, which states that $n \propto a^{\beta}$ with $\beta \approx 1$, i.e. urban population grows proportional to the urbanized area.[4,5]

Going to a lower scale, and entering the single city, a decay in the density of urbanized areas $\sigma(d;\, t)$ has been observed when increasing the distance d to the compact core of the city. This center with a high population density acts as "attractive center" for the city, and is usually called the *central business district* (CBD).[4] Real data suggest a dependence of the form $\sigma(d;\, t) \propto e^{-\lambda(t)d}$.[3,6] The function $\lambda(t)$ in the exponential is also known to vary with time, at least in the first stages of formation of the urban center, until it acquires a more or less stable population. While the correlation in urbanized areas expressed through $\sigma(d;\, t)$ decays faster when the city starts to form, the measured values are smaller at later times. This decrease in $\lambda(t)$ is interpreted as a trend towards decentralization which parallels the growth of the city.[7] Finally, the morphology of the external perimeter of the urban boundary is known to be fractal, with a dimension D which has been evaluated between 1.2 and 1.4 in most cases.[4]

The ubiquitous appearance of those coherent macroscopic patterns in human demography has led to the introduction of several theoretical models aimed at understanding how cities grow and evolve in time. The earliest attempts are found in the book of H. A. Simon,[8] where a simple and general mechanism of multiplicative nature is presented with the aim of explaining the distribution of cities by population or that of scientists by number of papers published, among others. More recently, approaches based on continuous deterministic[9] or stochastic diffusion models,[10] diffusion limited aggregation (DLA),[3] kinetic particle diffusion,[11] or interaction among n particles[12] can be found in the literature. In spite of their obvious differences, all of them share a key assumption: *Cities attract people.* This is a feature that leads to a positive feedback in the dynamics of urban settlements. An additional property is also involved: City growth takes place on a two-dimensional spatial domain where diffusion- and aggregation-like processes occur.

2. REACTION-DIFFUSION MODEL: EQUILIBRIUM PROPERTIES

With the previously listed observations at hand and aiming at recovering those universal large scale patterns, a new and simple reaction diffusion model for city formation is introduced.[13] The study of the stationary properties of the model returns distributions for several measures which compare very well with field observations, as we review in this section.

Our discrete conservative model runs on a square lattice of size $L \times L$ with periodic boundary conditions. Each cell at the position $i,\ j$ at time t has a total population $m(i,\, j;\, t)$. The following dynamical rules define the evolution of the model:

1. **Diffusion.** At each time step, each cell loses a fraction α of its contents, which is evenly distributed among its four neighboring cells:

$$m(i,\, j;\, t+1/2) = (1-\alpha)m(i,\, j;\, t) + \frac{\alpha}{4} \sum_{\langle k,l \rangle} m(k,\, l;\, t)\,, \quad (1)$$

where $\langle k,\, l \rangle$ indicates summation over the four nearest neighbors.

2. **Reaction.** With probability p, each cell multiplies its content by a factor p^{-1}, and with the complementary probability, $1-p$, its resources are set to zero:

$$m(i,\, j;\, t+1) = p^{-1} m(i,\, j;\, t+1/2) \quad \text{with probability } p\,,$$

$$m(i,\, j;\, t{+}1) = 0 \quad \text{with probability } (1{-}p)\,. \quad (2)$$

This model leads to the scaling law $f(m) \approx m^{-z}$ with $z = 2$ irrespectively of the parameters α and p. This can be theoretically understood by assuming that a stationary distribution $f(m)$ exists and that the global process is mainly led by the reaction step. In fact, under these assumptions, the new distribution $f'(m')$ of the rescaled variable $m' = m/p$ reads

$$f'(m')dm' = \frac{1}{p} f'\left(\frac{m}{p}\right) dm = pf(m)dm \quad (3)$$

and in the stationary regime, $f = f'$. In this case, the previous identity is fulfilled by the function $f(m) \propto m^{-2}$. This is our first result in which the chosen parameters play no role, supporting the universality of the suggested mechanism.

The exact translation of this result to real demography is that the distribution of population in *equal areas* would follow a power law with exponent -2. In Fig. 1, we display measures of the distribution of sizes according to population for the 100 largest countries of the world,[14] for 10 countries of South Europe[14] and for the 1300 administrative divisions of Switzerland,[15] and then compare them with the well-known distribution of city sizes (the 2700 largest in the world[14]). For the data of countries in South Europe, we have divided the total surface in equal areas of 10 km^2. According to such partition, a city like Berlin would contribute with about 10^2

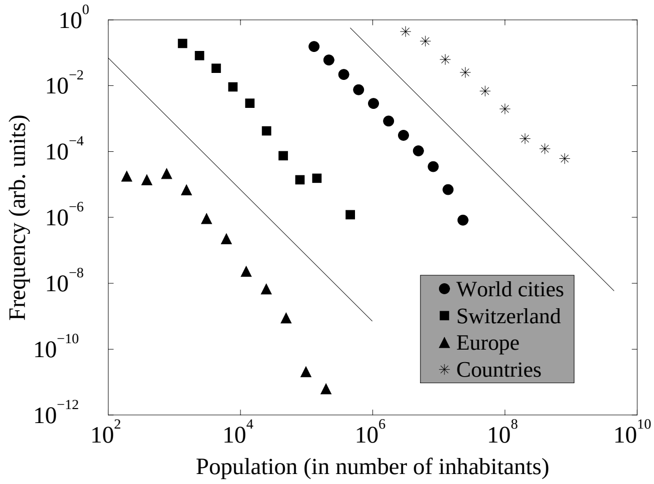

Fig. 1 Histograms of country and Swiss municipalities sizes according to population and of occupation number of equal areas, compared with the well-known city size distribution. For the sake of clarity, the data sets have been mutually shifted in the vertical direction. The straight lines have slope -2.

points to this last distribution (it is just one event when whole settlements are considered). While the distribution of the largest cities translates into the "classical" Zipf's law (when cities are ordered according to their rank), the first three measures represent an extension of it to any populated area.[16] Our data seem to point to the invariant character of demographic distributions no matter how they are defined. From the equal area partition of Europe to the size of the largest countries (with data spanning six orders of magnitude), we obtain distributions compatible with $f(n) \propto n^{-2}$. Least squares fit to every data set return $z = 1.73(7)$ (countries), $z = 2.16(11)$ (Swiss municipalities), $z = 2.17(18)$ (South European countries), and $z = 2.01(1)$ (largest cities of the world).[17]

In our model, the parameters p and α would describe different social conditions (economy, urban planning, rapid movements from or towards cities and so on). One would associate values of p close to 1 and of α close to 0 to areas with an old urban tradition and close to saturation (like West Europe). Cities in formation, with a higher interchange and still absorbing population from outside would be better represented with lower p and higher α (this is the present situation in Africa, for instance). In the latter case, some other terms should probably be added, as considered in the next section. In the most general case, the parameters p and α should vary with time, in a complex deterministic or in a stochastic fashion. Let us take the most extremal case and suppose from now on that our parameters are chosen at random *at each time step*, $p \in [0.5, 1]$ and $\alpha \in [0, 1]$. The results in this section have been obtained under this assumption.

With this modified model of wider applicability, we can jump to a larger scale. The multiplicative stochastic process together with diffusion leads to the appearance of spatial intermittency, of complex clustered patterns. Although the lifetime of a single cell can be estimated as $1/p$, the presence of neighboring cells able to lose a part of their contents by diffusion maintains the population in those sites affected by reaction. Therefore, large clusters of connected sites are transformed into persistent, long-lived entities, which we identify with cities in the following.

The first properties to be studied are the distribution $f(a)$ of areas covered by our virtual cities and that of the total population in each cluster, $f(n)$, as well as the relationship between these two quantities. The area a is the number of occupied connected sites in the cluster, and the total population will be $n = \sum_a m(i, j)$, where the sum is performed over this area. Supposing further that a and n are dependent quantities, a simple change of variables gives us the relation between the exponents r, s and β,

$$f(a)da = f(n)dn$$

and simply integrating we obtain $a^{-s+1} \propto n^{-r+1}$. Thus,

$$\beta = \frac{-s+1}{-r+1}\,. \tag{4}$$

The direct measurement of s^3 and r predicts $\beta \approx 0.9$, and field measures give $\beta \approx 1$.[4,5] From our numerical results, $s = 1.93(3)$ and $r = 1.90(3)$, we thus would expect $\beta \approx 1.03$. The dependence between n and a returns $\beta = 1.03(3)$ (see Fig. 2). These numerical results are suggestively close to the integer values $r = s = 2$, $\beta = 1$.

Let us now consider the decay in the urbanization density $\sigma(d; t)$ of actual city systems, which is known to follow an exponential decay $\sigma(d; t) = \sigma_0 \exp[-\lambda(t)d]$, where d is the distance to the CBD. In our simulations, we identify the CBD with the most populated cell in the lattice, which occupies the position $\mathbf{x}^*$,

$$m(\mathbf{x}^*, t) = \max\{m(\mathbf{x}, t)\}\,.$$

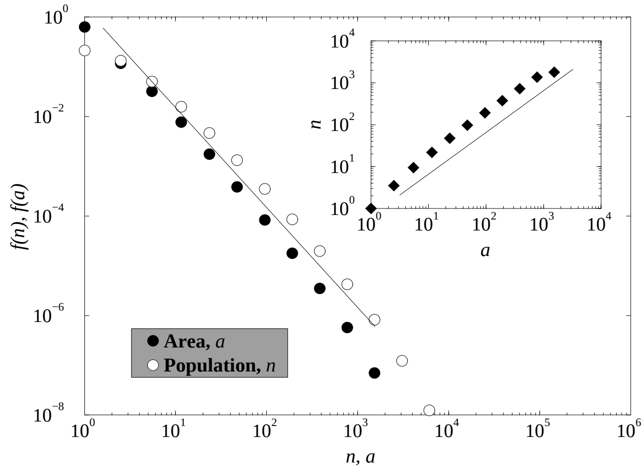

Fig. 2 Distribution of city population $f(n)$ and city areas $f(a)$ for random varying parameters in the reaction diffusion model. The continuous line has slope -2. Least squares fitting to the numerical data give $r = 1.90(3)$ and $s = 1.93(3)$. The relation between area and population is depicted in the inset. From those results, the model predicts that the number of inhabitants of a city grows proportional to its area [the numerical exponent for this relation is $\beta = 1.03(3)$]. The continuous line in the inset has slope 1. Real observations are in agreement with this result.

The formal expression for $\sigma(d; t)$ is

$$\sigma(d; t) = \frac{1}{2\pi d} \sum_{d=|\mathbf{x}^*-\mathbf{x}|} \Theta[m(\mathbf{x}^*, t)]\,, \tag{5}$$

where $\Theta(m)$ is the Heaviside step function.

We recover this exponential decay, which is directly related to the diffusion process in our model. Numerically, one obtains $0.030 < \lambda < 0.005$ in inverse lattice units. The real observations report variations in λ of at least one decade. In the next section, we will specify further the temporal variation of $\lambda(t)$.

The last observation to be checked is the geometry of the urban boundary. Extensive analysis has reported it to be fractal with a dimension $1.2 < D < 1.4$.[4] We have used an ordinary box-counting algorithm to estimate the fractal dimension, according to which $D = -\ln(N(l))/\ln(l)$, where $N(l)$ is the number of boxes of characteristic length l needed to cover the set (the boundary in the present case). A systematic study of many different realizations of the model returns values of the fractal dimension $1.15 < D < 1.35$, the higher the dimension the larger the cluster on the average, also in agreement with real observations.[4] Some empty areas in the interior of the city are always observed:

The clusters are non-compact objects and have fractal dimension below 2.

3. GROWTH OF CITIES: TRANSIENT PROPERTIES

As already discussed, migration plays the main role in the first stages of city formation. We can easily imagine that an initial site acting as a seed and furnished with some interesting properties (new jobs, more resources), acts as an attractor for the population outside this small but rich domain. When new people arrive at this settlement, they would try to place themselves as close to the seed as possible, producing in this way an extension of the populated region to adjacent sites. From time to time, they may also ground some small settlements near but not adjacent to the original one. Simultaneously to the arrival of new population, the whole system redistributes and reorganizes, producing dynamic exchanges among already urbanized sites.

We can modify the model discussed in the previous section in order to account for the formation of the city. Although different realizations of the process are possible, we present the following one as a paradigmatic example:

1. At every t and with a small probability p_n, an empty site chosen at random receives a population Δm.
2. For every t, a number Q of urbanized cells is chosen at random and supplied with an additional population Δm.
3. At every time step and after applying rules 1 and 2, the total population is redistributed according to the reaction diffusion process discussed in the previous section.

The particular choice of our parameters Q, Δm and p_n does not affect the statistical properties of the system as long as the addition of new population and new urbanized sites to the system is slow enough. More specifically, if we consider the total mass $N(t) = \sum_{i,j=1}^{L} m(i,\,j;\,t)$, a slow addition of population means $Q\Delta m \ll N(t)$. This condition is only violated at the very initial time steps, and will always be fulfilled for large t. For the addition of new sites disconnected from the bulk (that is, those sites connected to the seed that appeared first), we have to demand $1 \gg p_n \simeq 0$. We remark that the degree of competitivity of a new site to act as a seed for a new urban center decreases (for fixed p_n) as time elapses, because there will typically already be a large settlement in the system (one at least) which will absorb the main part of the new added population.

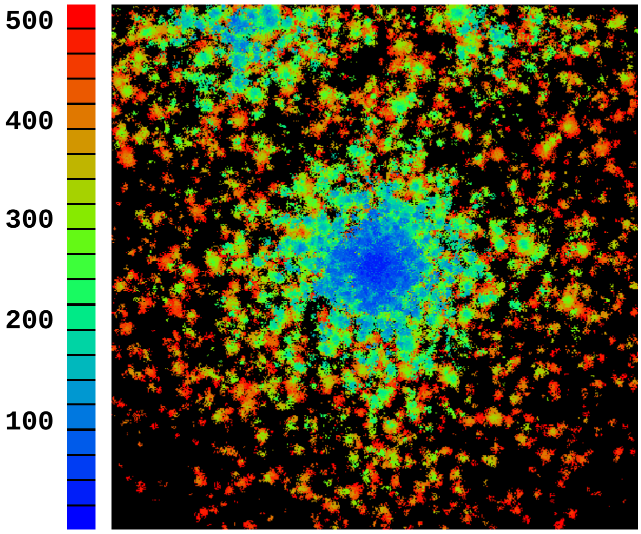

Fig. 3 Growth of a city. We start with a seed placed at the center of the lattice and slowly add population to the occupied cells. From time to time, new disconnected settlements are introduced. The color scale indicates the time at which every urbanized site enters the system. There is a clear morphologic contrast between the old center and the already developing surroundings.

Let us study the transient dynamics of this model for the set of parameters $Q = 10$, $p_n = 0.02$, and Δm a random number drawn from a uniform distribution between 1 and 100. In this section, we fix $p = 0.75$ and $\alpha = 0.25$ for the reaction diffusion step. Figure 3 represents a typical realization with the previous parameters on a lattice of size $L = 500$ and open boundary conditions. For visualization purposes, we have chosen the central site to be the initial seed, and have represented the subsequent evolution of the urbanized area in time. Every 25 time steps, we localize the new sites in the system. They are assigned the next color (starting with deep blue) until $t = 500$. We observe a rough boundary developing around the old center, and new urbanized centers appearing outside the borders of the main cluster.

Both the total urbanized surface and the total population increase as time elapses. Although the new population is added in a more or less regular fashion, we have found that it organizes in such a way — under the dynamical rules of the model — that the total population grows proportionally to time, $N(t) \propto t$. It saturates for long enough

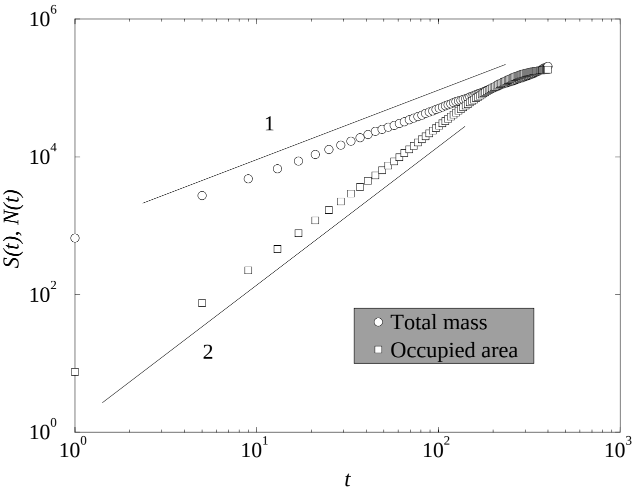

Fig. 4 Growth in the total mass and in the occupied area for a forming urban center on a lattice of size $L = 500$. Each point is the result of averaging over 100 independent realizations. See main text for the remaining parameters. Solid lines have slopes 1 and 2 as shown. Least square fitting to the numerical data up to $t = 100$ give $N(t) \simeq t^{0.97(1)}$, $S(t) \simeq t^{1.96(1)}$.

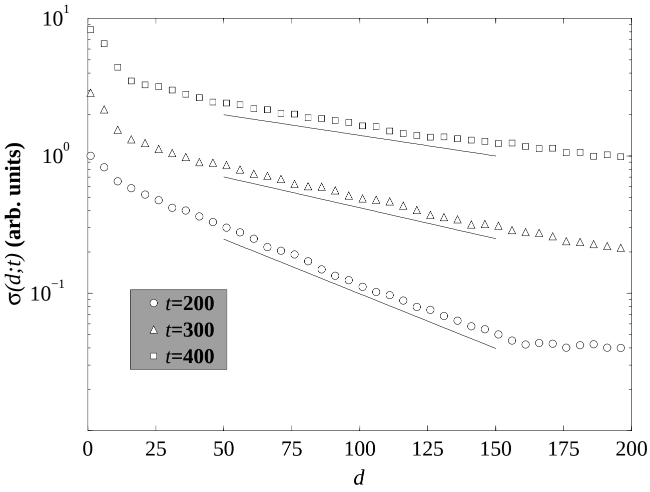

Fig. 5 Variation in the exponential decay for the spatial correlations as time increases. Empty symbols are numerical data for the same parameters as Fig. 4. Solid lines represent best fits to the data in the interval $50 \leq d \leq 150$. The two upper curves have been vertically shifted for better visualization.

t, and then large amplitude fluctuations appear as a result of the spatial intermittency caused by the reaction step (and maintained through diffusion).[13] The net growth of $N(t)$ proceeds until the addition of new population balances the loss through the boundaries. The total urbanized area is

$$S(t) = \sum_{i,j=1}^{L} \chi(i, j; t),$$

where $\chi(i, j; t) = 1$ if $m(i, j; t) > 0$ and 0 otherwise. It is observed to grow proportional to the square of time, $S(t) \propto t^2$, until its saturation value $S(\infty) = (1 - p)L^2$, around which Gaussian fluctuations are observed. When both $N(t)$ and $S(t)$ attain their equilibrium values, the system has reached a state of dynamic equilibrium completely equivalent to the one described in Sec. 2. Numerical results for population and area growth with the previous parameters and averaged over 100 initial conditions are represented in Fig. 4.

We are also interested at studying the variation of the urbanization profile from the CBD. As previously, we identify at each time the site with the largest population and use it as a representation of the compact center. Figure 5 displays the function $\sigma(d; t)$ at three different fixed times during the transient averaged over 100 independent realizations. The value of $\lambda(t)$ decreases with time, in a

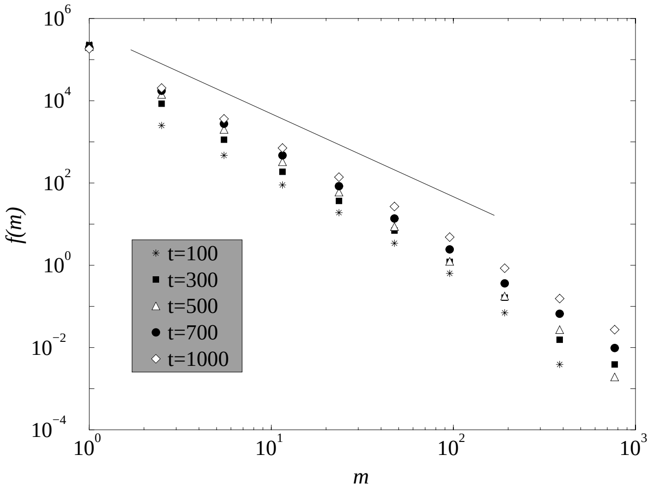

Fig. 6 Distribution of occupation values during the development of a city. The profile of $f(m; t)$ at different instances of time is compatible with the functional dependence $f(m) \propto m^{-2}$ (solid line), which is known to be a property of the equilibrium state.

way qualitatively equivalent to that observed in real city systems.[3] Least squares fitting of our numerical data give $\lambda(200) = 0.0182(1)$, $\lambda(300) = 0.0103(1)$ and $\lambda(400) = 0.0068(1)$. We would like to know if, during this transient, the distribution of occupation values $f(m; t)$ already displays the equilibrium profile. Considering that it is not easy to evaluate the stationarity of real city systems, and that in fact many of them are under formation (thus in the transient), it would still be desirable to obtain $f(m;t) \simeq m^{-2}$. Numerical results for $f(m;t)$ at different times during the transient are represented in Fig. 6. We observe a lack of very dense sites, to be ascribed to the relevant role played by diffusion when the urbanized area is comparatively small and actively expanding. Nevertheless, we still obtain, even for short times, a distribution almost indistinguishable from its stationary form (continuous line in Fig. 6).

4. CONCLUSIONS

Despite the great differences existing in the developing processes giving rise to each particular city, some large scale features have been identified worldwide and observed to be shared among very different metropolis. This surprising universality of the city size distribution (according to population or area) or of the two-d correlations in urbanized sites has led us to suggest a simple reaction diffusion model containing in our view only the essential ingredients of city formation and demographic transport. In fact, the results contained in this paper strongly support the conjecture that a single and simple mechanism based on a stochastic reinforcement rule, together with spatial diffusion, controls the large scale dynamics of urban growth. This is also in complete agreement with previous evidence stating that the basic mechanism of urban development is multiplicative in nature, and that an absolute increase in the total population does not alter the profile of the city size distribution.[2] A good consensus with the data for city areas distribution, for the population-area law, for the exponential decay in urbanized areas from the CBD, and for the fractal dimension of the boundary of urban systems has also been obtained.[13] The reaction diffusion model for demographic transport has been extended in order to account for the dynamic formation of a city. Our model has been successfully compared also to real data describing this transient behaviour. Some further modifications might be added in order to take into account geographical constraints, for instance. These could be represented as irregular boundaries corresponding to coasts and mountain chains, or one may assign different values to the probability of urbanization of different sites. Those changes would probably modify the morphology of the urban center, although its statistical properties should be maintained. Moreover, one could consider how the construction of a highway changes the described patterns. Fast ways connecting urban centers add a sort of ballistic motion to the slower diffusive transport, which intuitively should modify the average lifetime of our virtual metropolis.

Previous models of urban growth involved multiparametric characterizations of interactions between "agents" which cluster together and/or move in space.[18] Most detailed models include a set of rules which emphasize the economy-dependent nature of urban settlements. Such models are usually applied to specific situations or remain highly qualitative in their basic conclusions. In this context, our results suggest that very generic mechanisms are at work and that a well-defined theory can be formulated for urban growth phenomena at the large scale level. But even at the smaller, single city scale, our model is able to reproduce a number of characteristic features, again interpretable in terms of generic mechanisms. Since the model is based on a spatiotemporal intermittency mechanism, path-dependent phenomena (with some characteristic quantitative properties) will be at work.[19] This implies that the long term growth of a particular domain (like say, a given non-developed area in the urban perimeter) will be partially determined by neighboring resources (i.e. by the diffusion term in our model). But the reaction term, responsible for the intermittent behavior, together with the previous historical events, will generate path-dependent effects. Since much work is in progress trying to understand how cities grow and expand into previously empty areas, further studies should explore how the present model can provide some understanding of the underlying phenomena at this scale.

The results reported in this paper look robust and encouraging enough to believe that the real process should contain the basic rules used in our model as main forces shaping the large scale properties of urban settlements.

ACKNOWLEDGMENTS

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14. Source: National Center for Geographic Information and Analysis, Global Demography Project (http://ncgia.ucsb.edu/).
15. Source: Swiss Federal Statistical Office (http://www.admin.ch/bfs/).
16. The definition of "city" is not well-established, and every country uses a different one. Usually, a locality is defined as urban if its population is larger that a certain cut-off. This minimum is nevertheless not fixed and oscillates between 200 (e.g. Iceland) and 10^4 (e.g. Italy and Senegal).
17. Despite the previous results, we should be cautious when evaluating the role of man in defining frontiers and thus splitting or merging together natural concentrations of population. It might indeed produce systematic deviations from the law $f(n) \propto n^{-2}$, and even completely destroy it. In this sense, the distribution of country sizes and of Swiss municipalities has to be considered with some care: We find the evidence appealing, but are aware that some other mechanisms might be at play.
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